

Invariant matchings of exponential tail on coin flips in \mathbb{Z}^d

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Abstract. Consider Bernoulli(1/2) percolation on \mathbb{Z}^d , and define a perfect matching between open and closed vertices in a way that is a deterministic equivariant function of the configuration. We want to find such matching rules that make the probability that the pair of the origin is at distance greater than r decay as fast as possible. For two dimensions, we give a matching of decay $cr^{1/2}$, which is optimal. For dimension at least 3 we give a matching rule that has an exponential tail. This substantially improves previous bounds. The construction has two major parts: first we define a sequence of coarser and coarser partitions of \mathbb{Z}^d in an equivariant way, such that with high probability the cell of a fixed point is like a cube, and the labels in it are i.i.d. Then we define a matching for a fixed finite cell, which stabilizes as we repeatedly apply it for the cells of the consecutive partitions. Our methods also work in the case when one wants to match points of two Poisson processes, and they may be applied to allocation questions.

§1. Introduction.

Fix $(\Omega, \Sigma, \mathbf{P})$, where $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, Σ is the product σ -algebra, and \mathbf{P} is the product of Bernoulli measures with parameter 1/2. We prove the following theorems.

THEOREM 1.1. *For $d = 1, 2$, there exists a deterministic perfect matching $\phi_\omega = \phi$ between $\{x \in \mathbb{Z}^d : \omega(x) = 0\}$ and $\{x \in \mathbb{Z}^d : \omega(x) = 1\}$, such that for almost every $\omega \in \Omega$, ϕ is an equivariant function (i.e., $\phi_\omega = \phi_{g(\omega)}$ for every translation g of \mathbb{Z}^d), and for any $r > 0$,*

$$\mathbf{P}[\text{dist}(o, \phi(o)) > r] < \frac{c}{r^{d/2}}$$

with some constant c .

THEOREM 1.2. *Consider $d \geq 3$, and $\epsilon > 0$ arbitrary. Then there exists a deterministic perfect matching $\phi_\omega = \phi$ between $\{x \in \mathbb{Z}^d : \omega(x) = 0\}$ and $\{x \in \mathbb{Z}^d : \omega(x) = 1\}$, such that for almost every $\omega \in \Omega$, ϕ is an equivariant function, and for any $r > 0$,*

$$\mathbf{P}[\text{dist}(o, \phi(o)) > r] < C \exp(-cr^{d-2-\epsilon}).$$

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The bound in Theorem 1.2 can be slightly tightened, see Remark 3.30. We also have some ideas that could possibly remove the “ $-\epsilon$ ” from the bound, but at the cost of much extra complication. However, the correct magnitude of the exponent is not known: the only lower bound is the trivial $C \exp(-cr^d)$.

For $d = 2$ Theorem 1.1 is new; the best known result has been that of [S]. Note that for $d = 1, 2$, Theorem 1.1 is essentially tight by a theorem of [HP2], which says that for any matching rule ϕ , $\mathbf{E}[\text{dist}(o, \phi(o))^{d/2}] = \infty$ for these dimensions. For higher dimensions it was believed by Holroyd and Peres that there would be an exponential bound (see [HP2] and also [S]).

Our proofs rely on the following theorem, which is of independent interest. Informally, it claims that there is a sequence of coarser and coarser partitions for the space that are deterministic functions of the point configuration, and still most of the cells are (approximate) cubes with i.i.d. Bernoulli labels in them.

A subset X of \mathbb{Z}^d (\mathbb{R}^d) is called a **k -pseudocube**, if X contains some $[k/2] \times \dots \times [k/2]$ cube, and X is contained in some $2k \times \dots \times 2k$ cube (these two are referred to as the **volume condition**), and finally, if $|\partial X| \leq c_0 |X|^{\frac{d-1}{d}}$, with $c_0 = 2^{2d} + 2^d d$ (referred to as the **isoperimetry condition**). This choice for c_0 is rather arbitrary, and any greater constant could be used; in particular, it is clear that the intersection of \mathbb{Z}^d with a pseudocube of \mathbb{R}^d is a pseudocube itself, with a c_0 that is only worse by some constant factor.

THEOREM 1.3. *Fix $o \in \mathbb{Z}^d$. There exists a sequence $\{\mathcal{Q}_i\}$ of coarser and coarser partitions of \mathbb{Z}^d that are equivariant functions of the configuration $\omega \in \Omega$, and such that there is an event A_i with $\mathbf{P}[A_i] \geq 1 - c2^{-2^{i-1}}$, such that conditioned on A_i , the following hold:*

- (i) *For each $j \geq i$, the cell C_j of o in \mathcal{Q}_j is a 2^j -pseudocube;*
- (ii) *conditioned further on the location of C_j , the labels in C_j are i.i.d. Bernoulli(1/2).*
- (iii) *There is some infinite subsequence $\mathcal{Q}_{\alpha(k)}$ such that if $i = \alpha(k)$ then C_i is a 2^i by 2^i cube.*

In fact the probability of A_i with the above properties can be made arbitrarily large by an appropriate choice of the parameters in our construction.

Theorem 1.1 is a straightforward corollary of Theorem 1.3.

Proof of Theorem 1.1. Given the sequence of partitions, the matching is defined similarly to [S]. Namely, consider the sequence \mathcal{Q}_i from Theorem 1.3, and as $i = 1, 2, \dots$, for each 2^i -pseudocube, match as many yet unmatched points as possible, each with a point of opposite label, but otherwise arbitrarily. The central limit theorem gives the claim. ■

A question similar to the ones above is when one considers a Poisson point process (the natural generalization of a set of uniformly distributed points in the unit cube to an

infinite domain), and colors each of the configuration points independently red or blue with probability $1/2$. This is the same as taking two independent Poisson point processes of the same intensity. Our goal in this setting again is to give an “optimal” perfect matching between the red and blue points, by some matching rule that is a deterministic and equivariant function of the random point set. (Informally, the matching rule is defined using the locations and colors of the configuration points, but no background information from the underlying space.) By an optimal matching we meant in the \mathbb{Z}^d case that the function $F(r) := \mathbf{P}[\text{distance of } 0 \text{ from its pair is } > r]$ decays as fast as possible. For the Poisson case, similarly, we condition on that 0 is a configuration point, and want to make the tail $F(r) := \mathbf{P}[\text{distance of } 0 \text{ from its pair is } > r \mid 0 \in \omega]$ tend to 0 fast. The setting of our question shows that requiring the matching rule to be invariant is natural, since it essentially means that $F(r)$ does not change if we replace 0 by any other point.

We will phrase and prove our theorems for the \mathbb{Z}^d case, but our methods can be easily adjusted to the matching problem for Poisson point processes. Furthermore, a sequence of partitions as in Theorem 1.3 can be obtained as an equivariant function of *one* Poisson point process, by the natural modifications of our proof. Because of possible applications, let us state this separately.

THEOREM 1.4. *Fix $o \in \mathbb{R}^d$. There exists a sequence $\{\mathcal{Q}_i\}$ of coarser and coarser partitions of \mathbb{R}^d that are equivariant functions of the configuration ω of a Poisson point process on \mathbb{R}^d , and such that there is an event A_i with $\mathbf{P}[A_i] \geq 1 - c2^{-2^{i-1}}$, such that conditioned on A_i , the following hold:*

- (i) *For each $j \geq i$, the cell C_j of o in \mathcal{Q}_j is a polyhedron and a 2^j -pseudocube;*
- (ii) *for any Lebesgue measurable $A \subset \mathbb{R}^d$, $\mathbf{P}[\omega \cap A \cap C_i \mid C_i] = \text{Poisson}(|A \cap C_i|)$.*
- (iii) *There is some infinite subsequence $\mathcal{Q}_{\alpha(k)}$ such that if $i = \alpha(k)$ then C_i is a 2^i by 2^i cube.*

Here $\text{Poisson}(\lambda)$ denotes the Poisson distribution of intensity λ .

The question that we address was first asked by Holroyd and Peres in [HP2]. The best results were of order $r^{-s(d)}$, where $s(2) < 2/3$, and $s(d) < 2$ for any d , [S]. Similar, but considerably simpler matching problems were fully solved in [HPPS]; we will go into the details later in this introduction. Our question is related to two intensively studied families of problems. First, invariant measurable functions (**factors**) of point processes have been of interest from a statistical point of view (e.g. Palm processes, allocations; see [Th], [HL] and further references therein), from the optimization aspect (such as minimal spanning trees, [Ale]), and from a more general interest about how much information can be extracted from a point process ([HP1] being a seminal paper in the area). Close relatives

to the matching questions treated here are the so called **allocation questions**, when one has to assign disjoint sets of measure 1 to every point of a Poisson point process of intensity 1 so that this partition of \mathbb{R}^d is a factor. We will return to this later. The other related branch of problems, from a different direction, is finding an optimal matching between independently distributed points in a unit cube. While this latter field is almost fully explored (as a result of work by Ajtai-Komlós-Tusnády, Talagrand, Shor, Yukich; see [Y], [Ta] for surveys and a still standing challenging problem by Talagrand), the methods there do not seem to apply for our setting, because of the difficulties arising from the infinite setting and invariance.

While the matching rule we are looking for has to be a *deterministic* function of the configuration, one may relax this requirement and allow extra randomness. The additional freedom we gain this way is considerable. An example is that using extra randomness allows one to partition \mathbb{R}^d (\mathbb{Z}^d) to “nice” subsets (e.g. cubes) with independent labels in it: simply take $k\mathbb{Z}^d + v$ with $v \in [0, k]^d$ chosen uniformly. (This partition cannot be defined as a factor, by ergodicity.) A partition with these properties enables one to use local matching rules, repeated for countably many, coarser and coarser partitions.

A variant setting to our problem is when, instead of matching points of two colors, we have one Poisson point process, and want to find a perfect matching of optimal tail on its points. (For the \mathbb{Z}^d case the analogous problem is meaningless.) Call this a **1-matching problem** to distinguish from the **2-matching problem** defined earlier. Similarly to the matching question in a finite box, [Y], the 1-matching problem is much simpler than the 2-color case. The main reason for this is that much of the difficulties in the 2-color problem is coming from the difference between the number of vertices of the two colors within some given box. This discrepancy is around the square root of the number of points in the box, and it gives a lower bound to the number of points that cannot potentially be matched within the box. Since these points have to find their pairs beyond the boundary of the box, isoperimetry starts playing a role, and this is responsible for the dramatic change between dimensions ≤ 2 and ≥ 3 , as seen in Theorem 1.1 and Theorem 1.2. Obviously, the difficulty coming from discrepancies does not arise in the 1-color case.

Let us summarize briefly, what has been known about the four problems given by 1 and 2-color matchings, with randomized or deterministic matching schemes. See [HPPS] for a detailed account and an instructive table. The randomized 1-color matching has a sharp tail of order $\exp(-cr^d)$ for all dimensions. Similarly, for the randomized 2-color problem [HPPS] obtained a sharp $\leq cr^{-d/2}$ decay for $d = 1, 2$, and $\leq \exp(-cr^d)$ decay for $d \geq 3$. For the 1-color factor matching, the tight bounds are of order $\leq cr^{-1}$ for $d = 1$, and $\leq \exp(-cr^d)$ for $d \geq 2$. For the 2-color factor matching, the best known upper bound

was $r^{-s(d)}$ with $s(d) \leq 2/(1 + 4/d)$, [S].

Let us point at another interesting phenomenon about the decay rates of various problems. First, there is a big gap between the optimal decay rates of the randomized and the deterministic 1-color matching problems in dimension 1. Since the distance between two neighboring configuration points of a Poisson point process in 1 dimension follows an exponential distribution, it is rather the slow, linear decay for the factor matching that is surprising. It sheds some light on how restrictive the requirement of giving a *deterministic* perfect matching is.

Although Theorem 1.2 is a big improvement to earlier results, the degree of the optimal rate of decay is still open. A trivial lower bound is the following:

LEMMA 1.5. *For any 2-matching scheme on \mathbb{Z}^d*

$$\mathbf{E}[\exp(cr^d)] = \infty, \tag{1.6}$$

where r is the distance between the origin and its pair, and c is some positive constant.

Similar statement holds for the Poisson case, as mentioned before. For a proof observe that the distance of the origin to the closest point of opposite color is a lower bound for r , and this already satisfies (1.6).

Though allocation questions have a flavor similar to matching questions, we do not know of any direct connection that would make them equivalent in some sense. If ω_1 and ω_2 are the configurations of two independent Poisson point processes of intensity 1 (call them yellow and blue points), and A_i is a deterministic, invariant allocation rule ($i = 1, 2$), so that $A_i(x)$ is a set of measure 1 for each $x \in \omega_i$, and these partition \mathbb{R}^d for each i , then define a bipartite graph $\Gamma = (V, E)$; $V = \omega_1 \cup \omega_2$. Namely, let x_1 and x_2 be adjacent if $x_1 \in \omega_1$, $x_2 \in \omega_2$, and $A(x_1) \cap A(x_2) \neq \emptyset$. Then König's theorem (generalized to locally finite infinite graphs) implies that there is a perfect matching in Γ . Hence, as observed by Holroyd and Peres, if one could define a perfect matching for Γ in an *invariant* way, then one would get a matching rule between ω_1 and ω_2 that has essentially the same tail behavior as the allocation rule (at least for tails that decay relatively fast). Hence, the existence of an invariant perfect matching could be used to give a perfect matching from an allocation rule. The best known allocation rule so far is the so-called gravitation allocation, [CPPR]. Although the matching scheme presented in the present paper has a better tail than what is proved for the gravitation allocation rule in [CPPR], the following question is still of interest.

CONJECTURE 1.7. *Let $G = (V, E)$ be a random locally finite graph with $V \subset \mathbb{R}^d$, and such that for any isometry (translation) g of \mathbb{R}^d , (gV, gE) has the same distribution as (V, E) .*

Suppose that there is a perfect matching in G almost always. Then there is also a perfect matching that is a deterministic equivariant function of G .

The conjecture has a similar flavor to one asked by Bowen and Lyons, whether every quasi-transitive planar graph has a periodic 4-coloring, or a question of Lyons and Schramm, whether every infinite quasi-transitive graph has an invariant random coloring with as many colors as its chromatic number.

While Conjecture 1.7 leaves the question of creating matching schemes from allocation schemes open, the current proof for the matching question provided us with a tool for the allocation question. In joint work with Ander Holroyd, we are planning to apply the sequence of partitions in Theorem 1.4 to create an allocation rule using the technique in [AKT], which we believe may have a tight $\exp(-cr^d)$ tail.

A standard tool for the study of invariant processes is called Mass Transport Principle (MTP); see [HPPS] for a version that is close to our setting (Lemma 8), and also for further references. We will use the MTP via two of its straightforward corollaries, which we state separately:

LEMMA 1.8. *Suppose there is a given $\epsilon > 0$, a \mathcal{P} invariant partition of \mathbb{R}^d (\mathbb{Z}^d) to measurable sets, and an invariant measurable subset S of \mathbb{R}^d (\mathbb{Z}^d). If every $C \in \mathcal{P}$ satisfies $|C \cap S|/|C| \leq \epsilon$, then for any point o of \mathbb{R}^d (\mathbb{Z}^d) one has $\mathbf{P}[o \in S] \leq \epsilon$.*

LEMMA 1.9. *Let S be some random measurable subset of \mathbb{R}^d (\mathbb{Z}^d), and M be an invariant perfect matching on the points of ω . Then $\mathbf{P}[o \text{ or } M(o) \text{ is in } S] \leq 2\mathbf{P}[o \text{ is in } S]$.*

In Section 2 we present a sequence of partitions as in Theorem 1.3. In Section 3 we prove Theorem 1.2. Namely, we present a matching algorithm for Bernoulli labelled points in some *fixed cube* and then this is used to give the desired invariant perfect matching on \mathbb{Z}^d . We will repeatedly apply the algorithm for bigger and bigger cubes coming from the $\mathcal{Q}_{\alpha(i)}$, using the matching algorithm in a way that we only rematch vertices of smaller and smaller density.

Section 2 and Section 3 are independent, except for that Section 3 uses Theorem 1.3. Similar terms and notation may have different definitions in the two sections.

In the rest of the paper c and c' always denote positive constants depending only on d , and their values may change from line to line.

§2. The sequence of partitions.

In this section we construct the sequence of partitions of Theorem 1.3. Since we achieve this using consecutive sequences of partitions, a look at the summary of Remark 2.11 may facilitate the reader.

We shall think of \mathbb{Z}^d as embedded in \mathbb{R}^d . We shall define Voronoi tessellations of \mathbb{R}^d , and then other partitions based on that, always using some subset of \mathbb{Z}^d , chosen as a deterministic equivariant function of the labelling. In all these cases there is also an inherited partition for \mathbb{Z}^d , defined by the cells of the tessellation. The reason we prefer to partition \mathbb{R}^d is because it sheds light on how the proof works for Poisson point processes, and also because some geometric arguments are simpler to phrase this way. On the other hand it is clear that if we construct the desired sequence of partitions for \mathbb{R}^d , that gives rise to a partition for \mathbb{Z}^d as in Theorem 1.3. For any subset A of \mathbb{R}^d , we say that A is measurable, if it is Lebesgue measurable. Denote by $|A|$ its Lebesgue measure. On \mathbb{Z}^d the σ -algebra that we consider is the discrete one, and $|A|$ stands, as usual, for the number of elements in A . If \mathcal{P} is some partition, then the sets that it consists of are called the *classes* or **cells** of \mathcal{P} . When o is a point of \mathbb{Z}^d , the cell that contains o is denoted by $\mathcal{P}(o)$.

We say that a partition P is a **refinement** of partition Q , if any two elements in the same class of P are also in the same class of Q . If P is a refinement of Q , then Q is a **coarsening** of P . By the union of partitions P_i we mean their coarsest common refinement, that is, the partition where two elements are in the same class if and only if they are in the same class for each P_i . We denote this partition by $\vee P_i$. Finally, if \mathcal{P} is a partition, \mathcal{Q} is some set of pairwise disjoint subsets of \mathbb{Z}^d , then the finest common coarsening of \mathcal{P} and \mathcal{Q} is the partition defined by the equivalence relation where two elements are equivalent if they belong to the same class of \mathcal{P} , or the same set in \mathcal{Q} .

The proof of the next lemma is straightforward.

LEMMA 2.1. *If \mathcal{P}'_i is a sequence of coarser and coarser partitions of \mathbb{Z}^d , \mathcal{Q} is a set of pairwise disjoint subsets of \mathbb{Z}^d , and \mathcal{P}_i is the finest common coarsening of \mathcal{Q} and \mathcal{P}'_i , then \mathcal{P}_i is a sequence of coarser and coarser partitions.*

In this section **balls** are understood in the infinity norm (except for two places, where we refer to the “usual” ball as the *norm-2 ball*). That is, the ball $B(x, r)$ of radius r around a point $x = (x_1, \dots, x_d)$ is $\{(y_1, \dots, y_d) \in \mathbb{Z}^d : \max_i |x_i - y_i| \leq r\}$. Similarly, the r -**neighborhood** of a set $H \subset \mathbb{R}^d$ is the union of all the balls of radius r around some point of H . Hence the terms “cube” and “ball” stand for the same objects, unless otherwise mentioned.

We denote by $\text{Vol}(A)$ the volume of a d dimensional polyhedron A , and by $\text{Vol}_{d-1}(\partial A)$

the surface area of A . By a **path** in a graph we always understand a simple path, that is, a (finite) sequence of vertices with no repetitions, such that any two consecutive ones are connected by an edge.

If H is some discrete subset of \mathbb{Z}^d , let $\mathcal{V}(H)$ be the Voronoi tessellation of \mathbb{R}^d determined by H . Given a cell of some Voronoi tessellation $\mathcal{V}(H)$, we call the (unique) element of H in the cell the **centre** of the cell.

Denote by S_k the set of points $x \in \mathbb{Z}^d$ with the property that any vertex y such that $\|x - y\|_\infty = i$, $i \in \{0, 1, \dots, k\}$, satisfies $\omega(y) = (-1)^i$. Given $x \in \mathbb{Z}^d$, call the configuration on the set $\{y : \|x - y\|_\infty \leq k\}$ a **k -bulb of x** , if $\omega(y) = (-1)^i$ whenever $\|x - y\|_\infty = i$, $i \in \{0, 1, \dots, k\}$, i.e. if $x \in S_k$. Thus S_k is defined as an equivariant function of the random labelling. A simple but important consequence of the definition is that $\cup_{n=k}^\infty S_n = S_k$ is “sparse”: the probability of being in S_k is 2^{-ck^d} . Another reason for the choice of S_k is that any two elements of S_k have distance at least $2k$ from each other, because the k -bulbs of two elements in S_k can intersect only in their boundaries. Hence in $\mathcal{V}(S_k)$ every Voronoi cell contains a norm-2 ball of radius k . The most important property of k -bulbs is stated in the next lemma.

LEMMA 2.2. *Let $t \in \mathbb{Z}^+$, o be a vertex of \mathbb{Z}^d . Let $B := B(o, t)$, and for a vertex $x \in \mathbb{Z}^d$ let $r_{t,x} := \max(2t, \text{dist}(x, o))$. Call x a t -giant (of the configuration ω), if the configuration on $B(x, r_{t,x}) \setminus B$ inherited from ω can be extended to $B(x, r_{t,x})$ so that we get an $r_{t,x}$ -bulb around x . Then*

$$\mathbf{P}[\exists t\text{-giant}] \leq Cc^{-t^d}$$

with some $C, c > 1$.

Proof. Clearly $|B(x, r_{t,x}) \setminus B| \geq \frac{|B(x, r_{t,x})|}{2}$. Hence the following hold:

$$\begin{aligned} \mathbf{E}[\text{number of } t\text{-giants}] &\leq \sum_{x \in \mathbb{Z}^d} 2^{-|B(x, r_{t,x}) \setminus B|} \leq \\ &\leq |B|2^{-|B(x, 2t)|/2} + \sum_{i=t+1}^\infty c_1 i^{d-1} 2^{-\frac{|B(x, i)|}{2}} \leq (2t)^d 2^{-2^{d-1}t^d} + \sum_{i=t+1}^\infty c_1 i^{d-1} c_2^{-i^d} \leq Cc^{-t^d}, \end{aligned}$$

with some constants C and $c_1, c_2, c > 1$. By Markov’s inequality the same upper bound is valid for the probability that there exists a t -giant. \blacksquare

Given some grid $v + r\mathbb{Z}^d, v \in \mathbb{Z}^d, r \in \mathbb{Z}$, define the **basic cubes of $v + r\mathbb{Z}^d$** to be the cubes that have the form $v + \{(x_1, \dots, x_d) \in \mathbb{R}^d : ra_i \leq x_i < r(a_i + 1)\}$ with some integers a_i .

For a $C \subset \mathbb{Z}^d$, denote by $\partial_\rho C$ the ρ -neighborhood of the boundary ∂C of C . Fix a point $o \in \mathbb{Z}^d$.

LEMMA 2.3. Let K be a convex polyhedron in \mathbb{R}^d , and K' be the union of basic cubes of $r\mathbb{Z}^d$ fully contained in K . If K contains a norm-2 ball of radius R around a point o , then

$$\frac{\text{Vol}(K \setminus K')}{\text{Vol}(K)} \leq \frac{cr}{R},$$

with some constant c depending only on the dimension.

Proof. Any point z of $K \setminus K'$ has some point of ∂K in its r -neighborhood, otherwise any $r \times \dots \times r$ cube containing z is contained in K , in particular $z \in K'$. Hence $K \setminus K'$ is contained in $\partial_r K$, and

$$\text{Vol}(K \setminus K') \leq \text{Vol}(K \cap \partial_r K) \leq r \text{Vol}_{d-1}(\partial K).$$

The last inequality is true by the convexity of K . On the other hand, $\frac{1}{d!} \text{Vol}_{d-1}(\partial K)R \leq \text{Vol}(K)$, which can be seen by subdividing K to pyramids with apice in o (similarly to [Ti]). Putting the two inequalities together gives the statement. \blacksquare

Given a subcube $H = \square_{i=1}^d [m_i, m_i + t]$ of \mathbb{R}^d , call hyperfaces of the form $\{x \in K : x_i = m_i + t\}$, with some i , **right faces**. Let \mathcal{P} be a partition of \mathbb{R}^d . Define $\theta(H)$ with respect to \mathcal{P} (\mathcal{P} treated as a hidden parameter of θ for simplicity) to be the union of \mathcal{P} -cells that are contained in H or intersect only right-faces of H . (See Figure 1 for an example.)

LEMMA 2.4. Let H be a $2^i \times \dots \times 2^i$ cube of the above form (i.e., with each 1-face parallel to some coordinate axis), $i' < i$, and \mathcal{P} be a partition of \mathbb{R}^d to convex cells C that satisfy one of the following:

- (i) C is a $2^{i'} \times \dots \times 2^{i'}$ cube with each 1-face parallel to some coordinate axis,
- (ii) C has diameter at most 1.

Then $\theta(H)$ satisfies $|\partial\theta(H)| \leq c|\partial H| = c'2^{i(d-1)}$ with constants c, c' depending only on the dimension.

Proof. Note that $\partial\theta(H)$ is contained in the boundary of some \mathcal{P} -cells each of which intersects ∂H . These cells are either $2^{i'} \times \dots \times 2^{i'}$ cubes (call their set S_1), or cells of diameter ≤ 1 (call their set S_2). Let $X \in S_1$ and $\delta(X) = \delta$ be the minimal number such that X intersects some δ dimensional face of H . Then $|X \cap H| \geq 2^{i'\delta}$, because the edges of X are parallel to the coordinate axes. Since the elements of S_1 are disjoint, we get that $|S_1| \leq \sum_{\delta=0}^{d-1} \frac{2^{i\delta}}{2^{i'\delta}} \times \text{number of } \delta \text{ dimensional hyperfaces of } H$. That is, $|S_1| \leq c2^{(i-i')(d-1)}$ with some c depending only on d . We obtain:

$$|\partial\theta(H)| \leq 2d2^{i'(d-1)}|S_1| + |\cup_{X \in S_2} \partial X| \leq c'2^{i'(d-1)}2^{(i-i')(d-1)} + |\cup_{X \in S_2} \partial X|.$$

For the second term here, we can use the crude upper bound $|\partial X| \leq 2^d$ for $X \in S_2$, and $|S_2| \leq |\partial H|$. Hence we have

$$|\partial\theta(H)| \leq c2^{i(d-1)} + 2^d|\partial H| \leq c|\partial H| = c'2^{i(d-1)}, \quad (2.5)$$

using that H is a $2^i \times \dots \times 2^i$ cube. ■

Fix sequences $\{a_i\}$ and $\{b_i\}$ to be $b_i := 2^{2^i}$ and $a_i := 2^{b_i}$.

Consider the refinement \mathcal{P}_k of $\mathcal{V}(S_{a_k})$ where we partition each cell C of $\mathcal{V}(S_{a_k})$ with center $v(C)$ using the basic cubes of $v(C) + b_k\mathbb{Z}^d$. That is, two points of \mathbb{R}^d are in the same class (cell) of \mathcal{P}_k , if they are in the same cell C of $\mathcal{V}(S_{a_k})$ and the same basic cube of $v(C) + b_k\mathbb{Z}^d$. By Lemma 2.3 and Lemma 1.8, we obtain:

LEMMA 2.6. *The probability that o is not in a $b_k \times \dots \times b_k$ cell of \mathcal{P}_k is at most $\mathbf{P}[o \text{ is in } \cup_{K \in \mathcal{P}_k} \partial_{b_k} K] \leq cb_k/a_k$.*

Next, let $\mathcal{R}'_j = \vee_{i=j}^{\infty} \mathcal{P}_i$. We will prove that:

LEMMA 2.7. *The probability that o is not in a $b_j \times \dots \times b_j$ cube of \mathcal{R}'_j is $\leq \sum_{k=j}^{\infty} cb_k/a_k + \sum_{i=j+1}^{\infty} c_0 b_j/b_i \leq c2^{-2^j}$.*

There are two possible reasons for $\mathcal{R}'_j(o)$ not to be a $b_j \times \dots \times b_j$ cube: either the $\mathcal{P}_j(o)$ already fails to be a $b_j \times \dots \times b_j$ cube (in this case we say that o is b_j -**bad**), or $\mathcal{P}_j(o)$ is intersected by some cell boundary of \mathcal{P}_i , $i > j$. The bounds for these two are provided by Lemma 2.6 and Lemma 2.8, and hence Lemma 2.7 will follow.

LEMMA 2.8. *Suppose $k \in \mathbb{Z}^+$, $\rho > 0$. The probability that the ball of radius ρ around o is intersected by some cell-boundary from \mathcal{P}_k is at most $c_0\rho/b_k$, where c_0 is some constant independent of ρ and k .*

Proof. It is clear that for any $b_k \times \dots \times b_k$ cell C in \mathcal{P}_k , $\text{Vol}(\partial_\rho C)/\text{Vol}(C) \leq c'\rho b_k^{d-1}/b_k^d = c'\rho/b_k$ with some constant c' , and thus Lemma 1.8 shows that the probability that $B(o, \rho)$ is intersected by some cell boundary is at most $\mathbf{P}[o \text{ is } b_k\text{-bad}] + c'\rho/b_k \leq c/2^{b_k} + c'\rho/b_k \leq c_0\rho/b_k$ with some constant c_0 , also using Lemma 2.6. ■

Proof of Lemma 2.7. By Lemma 2.8 and Lemma 2.6, the probability that o is in a cell of \mathcal{R}'_j that does not coincide with a $b_i \times \dots \times b_i$ cell of \mathcal{P}_i is at most $\mathbf{P}[o \text{ is } b_i\text{-bad}] + \mathbf{P}[\text{the } \mathcal{P}_i\text{-cell of } o \text{ is intersected by the boundary of some } \mathcal{P}_j\text{-cell, } j > i] \leq cb_i/a_i + \sum_{j=i+1}^{\infty} c'b_i/b_j$, with some constants c, c' . ■

For any j , call the cells of \mathcal{R}'_j that are not $b_j \times \dots \times b_j$ cubes **irregular** cells. Note that by Lemma 2.7, the probability that o is contained in an irregular cell of some \mathcal{R}'_k ,

$k \geq j$, is $\leq c'2^{-2^j}$. Hence every irregular cell (in any of the \mathcal{R}'_j) is contained in some maximal irregular cell of some \mathcal{R}'_k ($k \geq j$), and the probability that this k is greater than some κ is $\leq c2^{-2^\kappa}$. Let \mathcal{I} be the set of maximal irregular cells. We mention that \mathcal{I} is not necessarily a partition of \mathbb{R}^d , but a set of pairwise disjoint subsets of it. Let \mathcal{R}''_j be the common coarsening of \mathcal{R}'_j , and \mathcal{I} . By Lemma 2.1, \mathcal{R}''_j is still a sequence of coarser and coarser partitions.

Let \mathcal{R}_j be a refinement of \mathcal{R}''_j , to be defined as follows. For each cell C in \mathcal{I} , we subdivide C by a \mathbb{Z}^d grid (placed on C in some deterministic way, say with origin on an extremal point for some fixed hyperplane). Now replace every cell $C \in \mathcal{I} \cap \mathcal{R}''_j$ by this refinement. The other cells of \mathcal{R}''_j (those that are not cells of \mathcal{I}) are unchanged. Of course we still have a sequence of coarser and coarser partitions, and Lemma 2.7 remains valid for the resulting \mathcal{R}_j :

LEMMA 2.9. *The probability that o is not in a $b_j \times \dots \times b_j$ cube of \mathcal{R}_j is $\leq \sum_{k=j}^{\infty} cb_k/a_k + \sum_{i=j+1}^{\infty} c_0 b_j/b_i \leq c2^{-2^j}$. In this case, o is in a cell that is contained in a $1 \times \dots \times 1$ cube.*

An important fact is that the partition \mathcal{R}_j is completely determined by the elements of the S_i 's with $i = a_j, a_{j+1}, \dots$, and that by Lemma 2.9 a cell of \mathcal{R}_j is either a $b_j \times \dots \times b_j$ cube (in which case we call it a **good cell**), or a cell of diameter $\leq d^{-1/2}$. The \mathcal{R}_j satisfy the claim of Theorem 1.3 (as we show at the end of this section), except for that the sizes of the typical cubes grow fast (and not just double) as we increase j one by one.

Given a subcube H of \mathbb{R}^d , recall the definition of right-faces and $\theta(H)$ (with respect to some given partition of \mathbb{Z}^d) from before Lemma 2.4. Now we are ready to define the final sequence \mathcal{Q}_j of partitions, as in Theorem 1.3. The sequence $\{\mathcal{Q}_j\}$ will be such that $\{\mathcal{R}_i\}$ is a subsequence of $\{\mathcal{Q}_j\}$.

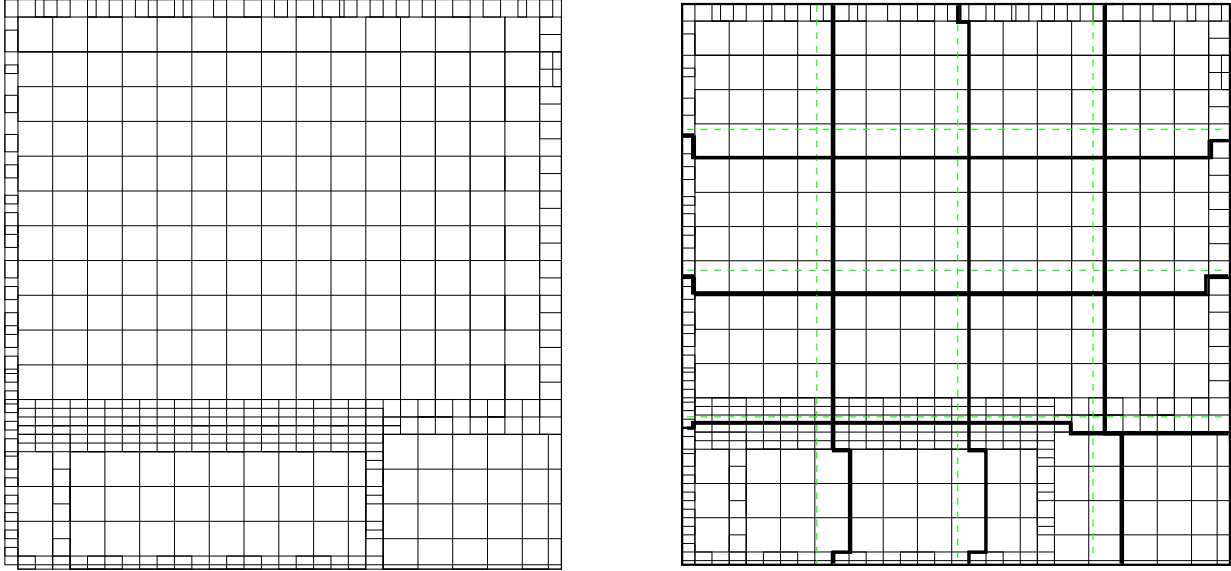
It is enough to define the ‘‘intermediate’’ partitions between \mathcal{R}_i and \mathcal{R}_{i+1} , for any i . Note that the cubic \mathcal{R}_i -cells do not necessarily subdivide C like a cubic grid, as illustrated by the left side of Figure 1. For each good cell C of \mathcal{R}_{i+1} (that is, a $b_{i+1} \times \dots \times b_{i+1}$ cube), and for $\ell = 0, 1, \dots, 2^i - 1$, consider the subdivision H_ℓ of C to dyadic cubes of size $b_i 2^\ell$. Now, for each $K \in H_\ell$, consider $\theta(K)$ with respect to the partition \mathcal{R}_i . Define \mathcal{Q}_i^ℓ as the set of $\theta(K)$'s as $K \in H_\ell$. (See the right side of Figure 1.)

LEMMA 2.10. *The cell C_o of o in \mathcal{Q}_i^ℓ is a $b_i 2^\ell$ -pseudocube whenever o is in a good cell of \mathcal{R}_{i+1} , unless $\ell = 1$ and o is in the b_i -neighborhood of the right boundary of its \mathcal{R}_{i+1} -cell. This exceptional event has probability $\leq b_i/b_{i+1} = 2^{-2^i}$.*

Proof. For $\ell = 0$ the claim is true even with cubes instead of pseudocubes. The volume condition for a pseudocube is clear from the construction and Lemma 2.3. To verify the

isoperimetry condition, let $H \in H_\ell$ be the cube that we used to define $C_o = \theta(H)$, and apply Lemma 2.4 together with the volume condition. \blacksquare

Finally, let (Q_j) be the sequence resulting from the finite sequences $(Q_i^\ell)_\ell$ when we put them one after the other as $i = 1, 2, \dots$



To the left: The subpartition of a good cell C of \mathcal{R}_{i+1} by the cells of \mathcal{R}_i . To the right: the partition H_ℓ of C by dyadic cubes (dashed), and the $\theta(K)$ (thick lines), $K \in H_\ell$.

Figure 1.

To finish the proof of Theorem 1.3, we shall show that for a fixed $o \in \mathbb{Z}^d$ the labels of the vertices in the Q_j -cell of o are i.i.d. conditioned on an event of probability tending to 1 rapidly with j .

Before going into that, let us remind ourselves to the construction of Q_j . See Figure 2a and 2b for schematic pictures of the sequentially constructed partitions and \mathcal{I} . Note that we changed the scales for the figure (and this is the cause of seemingly many noncubic cells, which is not the case when one uses the the proper parameters as in our construction).

REMARK 2.11. Summary of the construction of the partitions:

- (1) S_k consists of the vertices that have k -bulbs around them.
- (2) \mathcal{P}_k is a subpartition of the Voronoi tessellation $\mathcal{V}(S_{a_k})$ on S_{a_k} to $b_k \times \dots \times b_k$ cubes (with the exception of a small proportion of the cells).

- (3) \mathcal{R}'_k is the common refinement of the sequence of \mathcal{P}_i 's, with index i starting from k . This way most of the cells in \mathcal{R}'_k coincided with the cubes in \mathcal{P}_k ; on the other hand the \mathcal{R}'_k is a sequence of coarser and coarser partitions.
- (4) \mathcal{I} is the set of cells C that are not $b_k \times \dots \times b_k$ cubes, with $C \in \mathcal{R}'_k$, and maximal with this property. \mathcal{R}''_k is the common coarsening of \mathcal{I} and \mathcal{R}'_k .
- (5) We defined \mathcal{R}_k from \mathcal{R}''_k by subdividing its non-cubic cells (and possibly some others) to small chunks whose diameters are uniformly bounded. We still have a sequence of coarser and coarser partitions, and most of the cells are still b_k by b_k cubes.
- (6) Finally, \mathcal{Q}_j is a sequence that we obtained from \mathcal{R}_k by putting “intermediate” partitions in the sequence (\mathcal{R}_k) so that a “typical cell” is a pseudocube of size always doubling as j increases by 1. The subsequence \mathcal{R}_k provides the one given in (iii).

Let us point out that the constructions of the partitions $\mathcal{P}_j, \mathcal{R}_j, \mathcal{R}'_j$ and \mathcal{Q}_j did not use any information besides that coming from S_j, S_{j+1}, \dots :

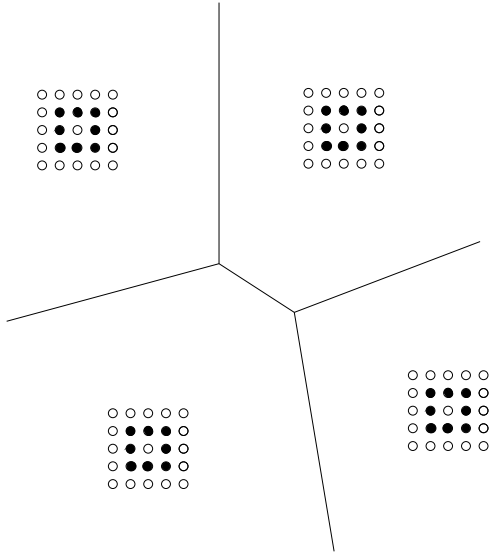
LEMMA 2.12. *Every partition \mathcal{Q}_j is a deterministic function of S_j .*

By Lemma 2.10 we know that \mathcal{Q}_j satisfies (i) in Theorem 1.3.

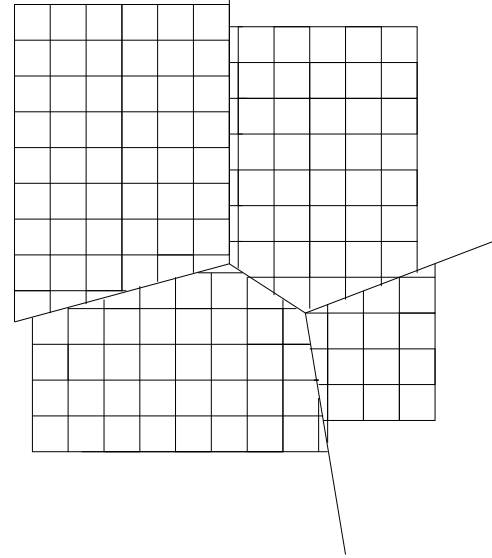
To show (ii) in Theorem 1.3, fix j and let i be such that $b_i < 2^j \leq b_{i+1}$, that is, $2^i < j \leq 2^{i+1}$. Let $B := B(o, 2b_{i+1})$. Note that the cell of o in \mathcal{Q}_j is contained in B . Further, a_i is such that one was using only elements of S_{a_i} (and hence possibly elements of S_n with $n > a_i$) to define \mathcal{R}_i (and thus \mathcal{Q}_j). Call a vertex x in \mathbb{Z}^d a **giant**, if it is an a_i -giant, as defined in Lemma 2.2 ($t = a_i$). Note that by definition, the existence of giants is independent of the configuration within B .

Our key observation is that if \mathbb{Z}^d contains no giants, then no element x of S_{a_i} can be so close to o as that the largest bulb around x intersects B . That is, if there are no giants, one can tell the elements of $S_{a_i}, S_{a_{i+1}}, \dots$ without looking into B . Hence the configuration outside B determines the cell C of o in \mathcal{R}_{i+1} , together with the subpartition of C by \mathcal{R}_i - and these two determine the cell of o in \mathcal{Q}_j . (Here we are using Lemma 2.12.) Then, conditioned on this event (no giants), the vertices in the cell of o in \mathcal{Q}_j have i.i.d. labels, since the cell of o is contained in B . Now, by Lemma 2.2, there are no giants with probability $\geq 1 - Cc^{-a_i^d}$. Furthermore, the probability that the cell of o in \mathcal{Q}_j is not a $2^j \times \dots \times 2^j$ pseudocube is at most the probability that it is not in a $b_i \times \dots \times b_i$ cube of \mathcal{R}_i plus the probability that it is at the right boundary of its \mathcal{R}_i -cell, as in Lemma 2.10. By Lemma 2.9 and Lemma 2.10, this is bounded by $c2^{-2^i} + c'b_i^{-1} \leq c''2^{-2^i} \leq c''2^{-2^{j-1}}$.

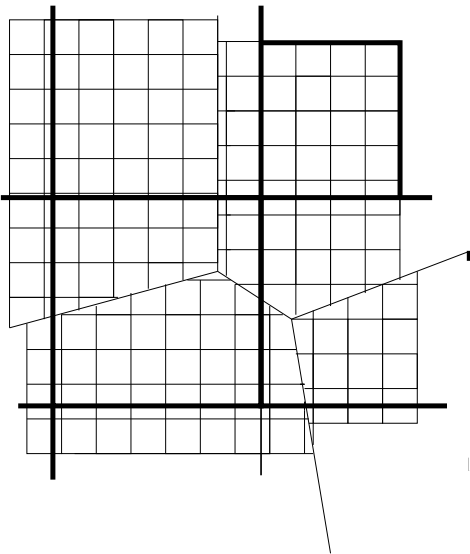
Define A_j to be the event that there are no giants, and the cell of o is a pseudocube. We have just seen that $P[A_j] \geq 1 - c2^{-2^{j-1}}$, and this finishes the proof of (i) and (ii) in Theorem 1.3. Part (iii) follows by setting $\mathcal{Q}_{\alpha(k)} := \mathcal{R}_k$.



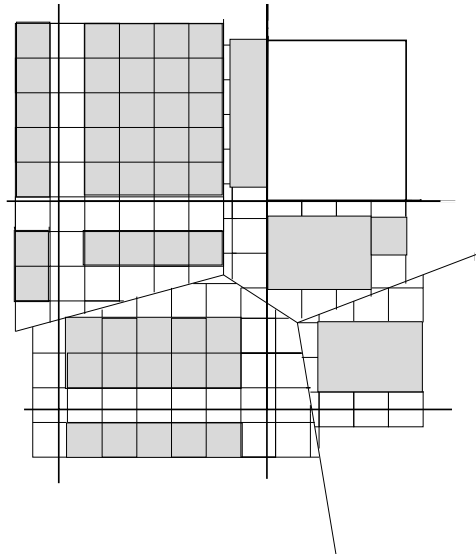
Four points of S_k , their neighborhood, and $\mathcal{V}(S_k)$



\mathcal{P}_k

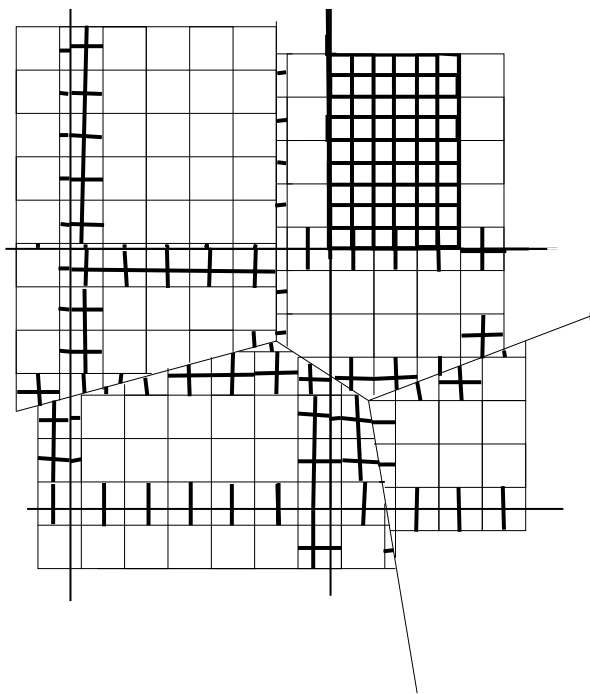


\mathcal{R}'_k (thicker lines indicate boundaries of \mathcal{P}_j , $j = k + 1, \dots$)



\mathcal{I} (the grey regions are not covered by any set of \mathcal{I})

Figure 2a. Construction of the sequence of partitions (on a scale altered from the real one)



\mathcal{R}_k , resulting from \mathcal{R}_k'' (the finest common coarsening of \mathcal{I} and \mathcal{R}_k') by subdividing irregular cells to cells of small diameters

Figure 2b.

§3. The matching rule.

Recall that an n -pseudocube or a pseudocube of size n in \mathbb{Z}^d is a subset C that contains some $n/2 \times \dots \times n/2$ cube and is contained in some $2n \times \dots \times 2n$ cube, and satisfies the isoperimetry condition. Call the elements of $\cup_i \mathcal{Q}_i$ **dyadic pseudocubes**. The reason for the name is that by Theorem 1.3, the pseudocube of \mathcal{Q}_i is about twice the size of the pseudocube in \mathcal{Q}_{i-1} , and these partitions are coarser and coarser, so most of the pseudocubes in \mathcal{Q}_i are subdivided by pseudocubes of \mathcal{Q}_{i-1} in a dyadic pattern.

For some subset S of \mathbb{Z}^d , denote by $y(S)$ the set of yellow elements of S , and by $b(S)$ the set of blue elements of S .

Given some $C \subset \mathbb{Z}^d$, with Bernoulli(1/2) labels on it, let the **surplus** of C be $|b(C) - y(C)|$. Denote this quantity by $\text{sur}(C)$.

Note the distinction between *subgraphs* of a graph G and *graphs on the vertex set of* G . Also, there is a bit of ambiguity about the use of the word *edge* : sometimes it refers

to edges of \mathbb{Z}^d , and sometimes to pairs in the matching, but it is always clear from the context.

From now on fix function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ to be $f(x) = x^{\frac{d-1-\epsilon/2}{d}}$. Say that a pseudocube C is **bad**, if $\text{sur}(C) > f(|C|)$.

Before presenting the matching rule, let us prove a few simple lemmas.

LEMMA 3.1. *Let $\Delta \subset \mathbb{R}^d$ be connected, and $U \subset \mathbb{R}^d$ with $\min_{x,y \in U} |x - y| \geq s$. Then if $\text{diam}(\Delta) < s$, then $|U \cap \Delta| \leq 1$. Otherwise*

$$|\Delta \cap U| \leq c|\partial\Delta|\text{diam}(\Delta)s^{-d}.$$

Proof. If $\text{diam}(\Delta) \leq s$, then the statement trivially holds by the assumption on U . So suppose $\text{diam}(\Delta) > s$. Let Δ_s be the s -neighborhood (“fattening”) of Δ . We have $|\Delta_s| \leq c|\partial\Delta|\text{diam}(\Delta)$. Since the balls of radius $s/2$ around points of $\Delta \cap U$ are disjoint, and they are all contained in Δ_s , we have $|\Delta \cap U| \leq c'|\Delta_s|s^{-d} \leq c|\partial\Delta|\text{diam}(\Delta)s^{-d}$. ■

LEMMA 3.2. *Let $S \subset \mathbb{R}^d$ be a cube, $S' \subset S$, $|S'| \geq |S|/2$. Then there exists an $S'' \subset S'$ with $|S''| = f(|S|) = |S|^{\frac{d-1-\epsilon/2}{d}}/2$ such that $\text{dist}(x, y) \geq c|S|^{\frac{1+\epsilon/2}{d^2}}$ for every $x, y \in S''$, with c depending only on d .*

Proof. Pick elements of S' for S'' one by one, always removing points of the $c|S|^{\frac{1+\epsilon/2}{d^2}}$ -neighborhood of the chosen point from S (and picking next elements from what remains). ■

Fix a k , to be determined later. We will choose it large enough, and a power of 2 for technical convenience. Note that in what follows, c and c' will always denote constants that *do not dependent on k* . Define $a(0) := \log k$, and $b(1) := 1$.

Choice of $a(i)$ and $b(i)$: Fix $b(i) > 2^{id}$, $i = 2, 3, \dots$, to be a subsequence of $\alpha(i)$ (where $\alpha(i)$ is as defined in Theorem 1.3). Fix an increasing subsequence $a(i)$ of $\alpha(i)$, $i = 1, 2, \dots$, such that $a(i) > \max\{b(i)^{2d/\epsilon}, \exp(\exp(i^{d-2}))\}$, and such that $\sum_{\nu=a(i)/d}^{\infty} (k2^\nu)^{-\epsilon/2} \leq \frac{1}{2}k^{\frac{1}{d-1}}b(i+1)^{-d}$.

The next lemma for later use is of elementary geometry. The claim is intuitively clear, but not trivial to prove. We do not need the specific value of the constant, but $c_d = 2^{d+2}$ works. Recall, that for any subset S of Z^d , we denote by ∂S the set of vertices with degree $< 2d$ in S . In the next lemma, if E is some set of edges and V is some set of vertices, let E_V stand for the subset of edges E incident to some element of V .

LEMMA 3.3. *There exists a constant c_d , depending only on d , such that the following holds. Let Γ be some subset of the edges of an $m \times \dots \times m$ cube K , and let $\Delta_1, \dots, \Delta_h$ be the*

connected components of $K \setminus \Gamma$. Then for all but at most one of the j 's we have

$$|\partial\Delta_j| \leq c_d |\Gamma_{\partial\Delta_j}|.$$

Consequently, there is a Δ_i such that

$$\sum_{j \neq i} |\partial\Delta_j| \leq 2c_d |\Gamma|.$$

Proof. Let i be such that $|\partial\Delta_i \cap \partial K|$ is maximal (in case of ambiguity, decide arbitrarily). Take any $j \neq i$. For $x \in \partial K$ and $z \in K$, define $P(x, z)$ as a path from x to z that makes as few turns as possible (otherwise its choice is arbitrary). Regard $P(x, y)$ as some element of the vector space $\mathbb{R} \vec{E}^{(K)}$, where $\vec{E}^{(K)}$ is some arbitrary fixed orientation of the edges of K . For each element (x, y) of $(\partial\Delta_j \cap \partial K) \times (\partial K \setminus \partial\Delta_j)$, let $P_{x,y} \in \mathbb{R} \vec{E}^{(K)}$ be $\frac{1}{|K|} \sum_{z \in K} (P(x, z) - P(z, y))$. Finally, define a flow μ as $\sum_{x,y} \frac{1}{|\partial K \setminus \partial\Delta_j|} P_{x,y}$, where the sum is over $(x, y) \in (\partial\Delta_j \cap \partial K) \times (\partial K \setminus \partial\Delta_j)$. It is not hard to see that the flow through any edge is $\leq \frac{(c_d-1)}{2} (1 + \frac{|\partial\Delta_j \cap \partial K|}{|\partial K \setminus \partial\Delta_j|}) \leq (c_d - 1)$ with some constant c_d , where the second inequality follows from $j \neq i$. (A proof for the 2-dimensional case, can be found in [A], Lemma 9. The only additional thing needed, is an upper bound of order $|K|$ for the number of paths $P(x, z)$, $(x, z) \in \partial K \times K$ containing an arbitrary edge e . Now, if $e = \{(v_1, \dots, v_\nu, \dots, v_d), (v_1, \dots, v_\nu + 1, \dots, v_d)\}$, then one of x and z has j 'th coordinate equal to v_j for every $j \neq \nu$, which gives an order $|K|^{1+1/d}$ choices for them, but also x has to be on the boundary, which makes the number of choices be of order $|K|$.) The total strength of the flow is $|\partial\Delta_j \cap \partial K|$. Since $\Gamma_{\partial\Delta_j}$ is a cutset between the sources and the sinks of the flow, the total amount flowing through it is equal to the strength. Putting this together with the bound on the amount flowing through an edge, we obtain:

$$|\partial\Delta_j \cap \partial K| \leq (c_d - 1) |\Gamma_{\partial\Delta_j}|.$$

Hence $|\partial\Delta_j| \leq |(\partial\Delta_j \cap \partial K) \cup \Gamma_{\partial\Delta_j}| \leq c_d |\Gamma_{\partial\Delta_j}|$. ■

The following lemma of several later uses is Chernoff's bound about independent Bernoulli sums, in the language of surpluses.

LEMMA 3.4. *The probability that a fixed pseudocube K of size $k2^i$ has surplus $\geq f((k2^i)^d)$ is $< \exp(-f((k2^{i-1})^d)^2 / 2(k2^{i+1})^d)$.*

Proof. Chernoff's bound provides an estimate $< \exp(\frac{-f(|K|)^2}{2|K|})$ for the probability. Then use the fact $(k2^{i-1})^d \leq |K| \leq (k2^{i+1})^d$. ■

Definition of the boundary of a partition: Given some partition \mathcal{P} of the vertices of some graph in \mathbb{Z}^d , we denote by $\partial\mathcal{P}$ the subgraph induced by the union of the (inner) vertex boundaries of the cells of \mathcal{P} .

Definition of \mathcal{P} : Consider \mathcal{Q}_i from Theorem 1.3. For a $C \in \mathcal{Q}_i$, let the cells of \mathcal{Q}_{i-1} in C be called the **bricks** of C . Let the cells of $\mathcal{Q}_{\log k} = \mathcal{Q}_{\alpha(0)}$ be called **basic pseudocubes**. For $j \geq \log k$, say that a cell $C \in \mathcal{Q}_j$ is **bad**, if $\text{sur}(C) \geq f(|C|)$, or if C is not a 2^j -pseudocube. This definition implicitly relies on j and \mathcal{Q}_j ; however, one can extend the definition of “bad” to any element C of $\cup_j \mathcal{Q}_j$ for the following reason. Since the \mathcal{Q}_j are coarser and coarser partitions, one can trace back for each $C \in \cup_{j=\log k}^{\infty} \mathcal{Q}_j$ the smallest j such that $C \in \mathcal{Q}_j$, and say that C is bad, if it is bad in \mathcal{Q}_j .

Call C **ripe**, if C is not bad, but one of its bricks is bad, and further, C is maximal among pseudocubes of $\cup_j \mathcal{Q}_j$ with this property with respect to inclusion. That in fact there is a maximal such C containing o for every o , follows from the next lemma.

LEMMA 3.5. *The probability that $\mathcal{Q}_{i+\log k}(o)$ is not bad, but one of its bricks is bad is at most $c \exp(-c'(k2^i)^{d-2-\epsilon})$.*

Proof. For $x \in C$, the probability that $\mathcal{Q}_{i-1+\log k}(x)$ is bad is bounded by $c \exp(-c'(k2^i)^{d-2-\epsilon})$. This is a consequence of Lemma 3.4 (in case $\mathcal{Q}_{i-1+\log k}(x)$ is bad because it has too large surplus), and Theorem 1.3 (in case $\mathcal{Q}_{i-1+\log k}(x)$ is bad because it is not a pseudocube). Hence

$$\begin{aligned} & \mathbf{P}[\mathcal{Q}_{i+\log k}(o) \text{ is not bad but some brick of it is bad}] \leq \\ & \leq \mathbf{P}[\mathcal{Q}_{i-1+\log k}(x) \text{ is bad for some } x \in B_{k2^{i+2}}] \leq c \exp(-c'(k2^i)^{d-2-\epsilon}) \end{aligned}$$

if k is large enough, for every $i \in \mathbb{Z}^+$. ■

A bound of the same magnitude holds for the probability of being in a ripe pseudocube as for the probability of being in a bad pseudocube of similar size, as stated in Lemma 3.6.

Call a pseudocube C an **elementary pseudocube** if it is ripe or it is a basic pseudocube that is not contained in any ripe pseudocube. It's a consequence of the definitions that elementary pseudocubes in $\cup_j \mathcal{Q}_j$ partition \mathbb{Z}^d ; call this partition \mathcal{P} . An important property of \mathcal{P} is that none of its cells $C \in \mathcal{Q}_j$ is bad in \mathcal{Q}_j . Lemma 3.6 gives an exponential bound on the tail probability of the diameter of the \mathcal{P} -cell of a point. In fact, the reason we defined pseudocubes, and the partitions of always doubling approximate cell sizes in Theorem 1.3 (instead of just taking the more convenient sequence $\mathcal{Q}_{\alpha(k)}$), is to have this control of the tail coming from Lemma 3.4 (which is inherited from the tail for bad cubes).

LEMMA 3.6. *The probability that a vertex is in a ripe pseudocube of size $k2^i$ is at most $c \exp(-c'(k2^i)^{d-2-\epsilon})$. Hence, the probability that the \mathcal{P} -cell of a point has size $k2^i$ has the same bound.*

Proof. By Lemma 3.4 and the fact that a ripe pseudocube of size $k2^i$ contains a bad pseudocube of size $k2^{i-1}$. \blacksquare

Definition of \mathcal{P}_i : If we find a maximal matching within each \mathcal{P} -cell, the set of unmatched points is “relatively small”. We will then match these points, using a sequence of coarser and coarser partitions. By definition, \mathcal{P} is such that for any cell C of \mathcal{Q}_i , $i \geq \log k$, C is either contained in some cell of \mathcal{P} , or it is a union of some cells of \mathcal{P} . Define \mathcal{P}_i as the common coarsening for \mathcal{Q}_i and \mathcal{P} , $i \geq a(0)$. (In particular, $\mathcal{P}_0 = \mathcal{P}$.) Since \mathcal{P} is a refinement of \mathcal{P}_i for any i , every cell of \mathcal{P}_i contains some elementary cell. In particular, we achieve the following, which was the goal of the last few paragraphs:

LEMMA 3.7. *If $i \geq \log k$ then no cell of \mathcal{P}_i is bad. That is, every cell $C \in \mathcal{P}_i$ is a $2^{i'}$ -pseudocube with some $i' \geq i$, and $\text{sur}(C) \leq f(|C|)$.*

Call a cell $C \in \mathcal{P}_j$ **cubic**, if it is a 2^j -cube, otherwise it is **non-cubic**. If $j = \alpha(i)$ for some i , then every cell of \mathcal{Q}_j that is not bad is cubic, hence $\mathcal{P}_j(o)$ can be non-cubic only if it is a cell in $\mathcal{P} \setminus \mathcal{Q}_j$, and hence $\mathcal{P}_j(o)$ has size $\geq 2^j$. This probability is bounded by Lemma 3.6. These two give us that with probability exponential in $2^{j(d-2-\epsilon)}$, a point is in a 2^j -cubic cell of $C \in \mathcal{P}_j = \mathcal{P}_{\alpha(i)}$.

For each i we will define an invariant monochromatic subset U_i of Z^d and a matching \mathcal{M}_i in such a way that $|U_i \cap C| = \text{sur}(C)$ for every $C \in \mathcal{P}_{\alpha(i)}$, and every vertex of $C \setminus U_i$ is matched by \mathcal{M}_i .

For every $2^{a(i)} \times \dots \times 2^{a(i)}$ cube S in $\mathcal{P}_{\alpha(i)}$, if the majority of points in S is blue, choose $S' \subset S$ to be the set of blue vertices, otherwise let S' be the set of yellow vertices. Choose S'' from S' as in Lemma 3.2. Finally, pick an arbitrary subset of cardinality $\text{sur}(S)$ from S'' ; call this $U_i(S)$. Now let $U_i := \cup U_i(S)$. (Of course the above choices can be made cell by cell in some predetermined way, that is the same for every translate of the cell, to make the resulting U_i invariant.) Note that

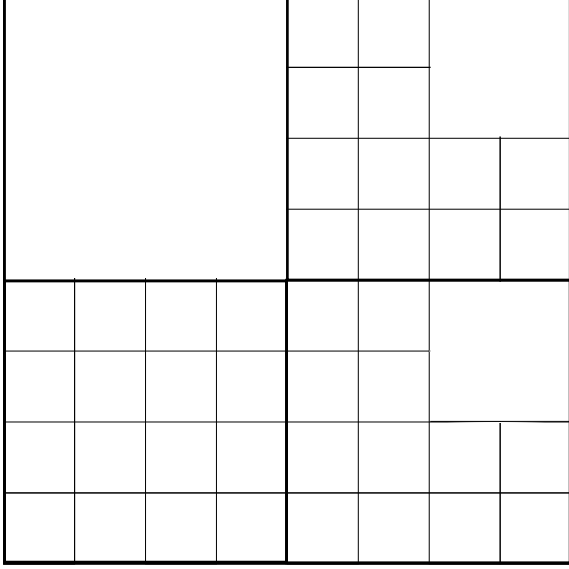
$$\min_{x,y \in U_i \cap S} |x - y| \geq c|S|^{\frac{1+\epsilon/2}{d^2}} = 2^{\frac{a(i)(1+\epsilon/2)}{d}}, \quad (3.8)$$

by our choice coming from Lemma 3.2.

Then we have, by Lemma 3.1, the following:

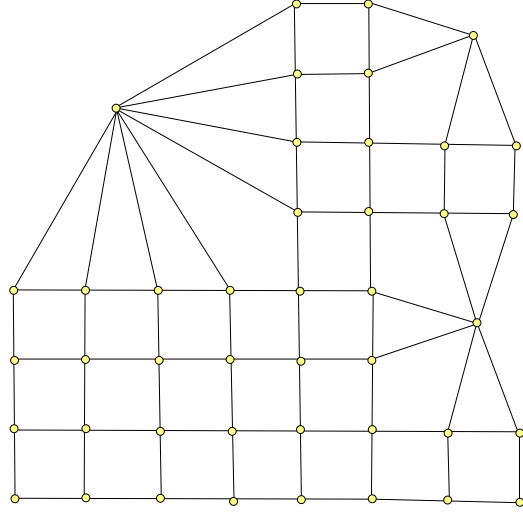
LEMMA 3.9. *For any S cubic cell of size $2^{a(i)}$ of $\mathcal{P}_{\alpha(i)}$, and connected subset Δ of S , either $|\Delta \cap U_i| \leq 1$, or one has*

$$|\Delta \cap U_i| \leq c|\partial\Delta|2^{a(i)}(\min_{x,y \in U_i} \text{dist}(x,y))^{-d} \leq c|\partial\Delta|2^{-\epsilon a(i)/2}.$$



$K \in \mathcal{P}_{a(i)}$

White cells represent cells of \mathcal{P} .
 Thick lines show cell boundaries of $\mathcal{P}_{b(i)}$.



\tilde{K}

The multiplicity of each edge is the number of edges between two small cubes on the left.

Figure 3. The contraction $\tilde{\beta}$ from K to \tilde{K} . (Scale altered from real. Pseudocubes of \mathcal{P} and $\mathcal{P}_{b(i)}$ are represented by cubes.)

Now we are ready to present the **matching algorithm**.

Step 1. For each elementary pseudocube C (i.e. cell of \mathcal{P}) match blue points with yellow points as long as it is possible.

Take the union of these matchings over all elementary pseudocubes C , and call the resulting matching \mathcal{M}_0 . Note that in each elementary pseudocube C the number of points not matched by \mathcal{M}_0 is equal to $\text{sur}(C) \leq f(|C|)$ (using Lemma 3.7 applied to $\mathcal{P}_{a(0)}$, i.e. the fact that an elementary pseudocube is not bad). Hence, at least $|C|/3$ vertices of each color in every elementary pseudocube $|C|$ are matched by \mathcal{M}_0 . This, and the bound on the number of unmatched points will have importance when we will define pairwise disjoint augmenting paths. It is also clear that \mathcal{M}_0 is invariantly defined.

The next lemma is straightforward from Lemma 3.6 and the definition of \mathcal{M}_0 .

LEMMA 3.10. *The probability that a vertex, given that it is matched by \mathcal{M}_0 , has its pair at distance $\geq r$ from it, is $\leq c \exp(-c' f(r^d)^2 / r^d) \leq c \exp(-c' r^{d-2-\epsilon})$.*

Denote by U_0 the set of points that are not matched by \mathcal{M}_0 .

For sets A and B , the *multiset* union of them will be denoted by $A \dot{\cup} B$; that is, $A \dot{\cup} B$ is the union of A and B , with elements in $A \cap B$ having multiplicity 2. If A is a multiset

and B is a set, then we define $A \cap B$ to be the multiset such that every x that is in B will have the same multiplicity in $A \cap B$ as in A .

In **Step 2**, we proceed in countably many stages. In **Stage i** , as $i = 1, 2, \dots$, we define a rematching procedure, that will match every point of $\mathbb{Z}^d \setminus U_i$ to some other point. In particular, it matches the points of $U_{i-1} \setminus U_i$ (which were unmatched before). The idea is that in Stage i we redefine only edges that have an endpoint in a set of \mathcal{P} -cells (which is “sparse” of density about $1/b(i)$). The method ensures that the tail of the matching remains of the same magnitude as for \mathcal{M}_0 (provided by Lemma 3.10). The rematching will use augmenting paths. When $i > 1$, in Stage i , the scarcity of U_{i-1} (which is a result of our choice for the sequence $a(i)$) makes the rematching simpler. However, in Stage 1, the set U_0 of points to be matched is coming from Step 1, and this fact is responsible for more difficulties and the sharp role of isoperimetry in this case.

Definition of $N(\tilde{K})$: For each cubic cell K in $\mathcal{P}_{a(i)}$ we will do the following. Fix i and a K in $\mathcal{P}_{a(i)}$ for the rest of this section. Define \tilde{K} by contracting every cell of \mathcal{P} to a vertex, that is, contracting elementary pseudocubes of K . We do not erase any edge after the contraction, multiple edges are allowed. Let $\tilde{\beta}$ be the contraction mapping from K to \tilde{K} . (See Figure 3.)

Then define $N(\tilde{K})$ to be a network on a graph (V, E) , with capacities on edges. Here V is the union of $V(\tilde{K})$ and two extra vertices B and Y , and E is the union of $E(\tilde{K})$ and all edges of the form $\{B, x\}$ or $\{Y, x\}$, $x \in V(\tilde{K})$. Define capacities as follows. Let every edge incident to some vertex in $\tilde{K} \setminus \partial\tilde{\beta}(\mathcal{P}_{b(i)})$ have capacity 0. Recall that $\partial\tilde{\beta}(\mathcal{P}_{b(i)})$ denotes the union of the vertex boundaries of the cells in the push-forward partition $\tilde{\beta}(\mathcal{P}_{b(i)})$. Let each edge induced by $V(\tilde{K}) \cap \partial\tilde{\beta}(\mathcal{P}_{b(i)})$ have capacity $k^{\frac{d}{d-1}}/12d'$, where $2d > d' \geq d$ is chosen so that it makes this number an integer (and at least 1, using that k is large enough). Note that for any $x \in \tilde{K}$, $\sum_{x \in e} \text{cap}(e)$ is at most half the number of points in $\tilde{\beta}^{-1}(\tilde{K})$ that are matched by \mathcal{M}_0 , by the definition of \mathcal{M}_0 and \tilde{K} , and the properties defining a pseudocube. The same will be true if we replace \mathcal{M}_0 by \mathcal{M}_{i-1} , because the endpoints of \mathcal{M}_{i-1} will always contain the endpoints of any previous matching. Finally, for every vertex x in $(U_i \dot{\cup} U_{i-1}) \cap K$, choose a “representative” $\bar{x} \in V(\tilde{K}) \cap \partial\tilde{\beta}(\mathcal{P}_{b(i)})$ such that x and $\tilde{\beta}^{-1}\bar{x}$ are in the same $\mathcal{P}_{b(i)}$ -cell. For each such \bar{x} , if x is blue and $x \in U_{i-1}$ or x is yellow and $x \in U_i$, add an edge of capacity 1 between B and \bar{x} , otherwise add an edge of capacity 1 between Y and \bar{x} . Let the other edges on B and Y have capacities 0. Note that the total capacity of the edges on B is equal to the total capacity of the edges on Y , by the choice of U_i (because $b(U_i \cap K) - y(U_i \cap K) = b(U_{i-1} \cap K) - y(U_{i-1} \cap K)$).

What we are really interested in is the set of vertices incident to some edge of nonzero capacity in $\partial\tilde{\beta}(\mathcal{P}_{b(i)}) \cup \{B, Y\}$, and hence the network induced by them. Having the bigger

network here, with many edges of zero capacity, has the advantage that it is easily related to K . This technical convenience will make it easy to use some of our geometric lemmas.

For an $X \subset \tilde{K}$, denote by $N_{B,X}$ and $N_{Y,X}$ the subnetwork of $N(\tilde{K})$ induced by $B \cup X$ and $Y \cup X$ respectively.

We will form a set $E(i, K)$ of pairs from the elements of $U_{i-1} \dot{\cup} U_i$ in K , and a set of paths $\tilde{\mathcal{P}}(i, K) := \{\tilde{P}(x, y) : \{x, y\} \in E(i, K)\}$, in such a way that:

- (i) elements of U_i are paired with elements of the same color in U_{i-1} , and all the elements of U_{i-1} that are not paired this way, are paired with an element of the opposite color in U_{i-1} . Call the set of such pairs $E(i, K)$. For any pair $\{x, y\} \in E(i, K)$, $\tilde{P}(x, y)$ is a path in $\tilde{K} \cap \partial\tilde{\beta}(\mathcal{P}_{b(i)})$ between \bar{x} and \bar{y} (where \bar{v} is defined from v as in the definition of $N(\tilde{K})$).
- (ii) Every vertex $\tilde{x} \in \tilde{K}$ is contained in at most $|\tilde{\beta}^{-1}(\tilde{x})|/3$ elements of $\cup_{i,K} \tilde{\mathcal{P}}$.

PROPOSITION 3.11. *If the maximal flow on $N(\tilde{K})$ from B to Y has $\sigma = \frac{|U_{i-1} \cap K| + |U_i \cap K|}{2}$ strength, then there exists a set of paths $\tilde{\mathcal{P}}(i, K)$ (with a set $E(i, K)$ for the pairs of endpoints), that satisfy (i) and (ii).*

If such $\tilde{\mathcal{P}}(i, K)$ exists, then there is a set $\mathcal{P}(i, K)$ of pairwise vertex-disjoint paths on the vertices of K such that for every $\tilde{P} \in \tilde{\mathcal{P}}(i, K)$, $\tilde{P} = (x_1, \dots, x_m)$, there is a $P \in \mathcal{P}(i, K)$, $P = (v_1, \dots, v_{m'})$, such that $v_1 \in U_{i-1}$, and:

- (I) $m' = 2m + 1$ if and only if $v_{m'} \in U_i$; $m' = 2m + 2$ if and only if $v_{m'} \in U_{i-1}$;
- (II) v_1 and $v_{m'}$ are such that $\tilde{\beta}\bar{v}_1 = x_1$ and $\tilde{\beta}\bar{v}_{m'} = x_m$;
- (III) for $\nu = 1, \dots, m$, one has $v_{2\nu} \in \tilde{\beta}^{-1}(x_\nu)$ and further, $v_{2\nu}$ and $v_{2\nu+1}$ are matched by \mathcal{M}_{i-1} . Moreover, the vertex colors along P are alternating.

Figure 4 illustrates the connection between the elements $\tilde{\mathcal{P}}(i, K)$ and $\mathcal{P}(i, K)$.

Proof. If there exists an admissible flow from B to Y of strength $\frac{|U_{i-1} \cap K| + |U_i \cap K|}{2} = \sum_x \text{cap}(\{B, x\}) = \sum_x \text{cap}(\{Y, x\}) = \sigma$, then there is also an integer valued flow, since the constraints on every edge are integers. Such a flow can be decomposed as a sum of paths. Delete the first and last edge (the ones incident to B and Y) from each of these paths, and define the set of resulting paths to be $\tilde{\mathcal{P}}(i, K)$. By the definition of $N(\tilde{K})$, this shows the first assertion, (i) and (ii).

We can take a preimage in K by $\tilde{\beta}$ for each element $\tilde{P}(x, y)$ in $\tilde{\mathcal{P}}(i, K)$, increase them by attaching one point of $U_{i-1} \cup U_i$ to each endpoint of the preimage, to find a set of pairwise disjoint alternating augmenting paths $P(x, y)$ that satisfy (I)-(III).

The connection between $P(x, y)$ and $\tilde{P}(x, y)$ is simply the following. The two endpoints of $P(x, y)$ are x and y . The $2j$ 'th point of $P(x, y)$ is going to be a vertex from $\tilde{\beta}^{-1}(x_j)$ that is covered by \mathcal{M}_{i-1} , and chosen to be of a color different from v_1 . Then

the $2j + 1$ 'st vertex will be the pair of v by \mathcal{M}_{i-1} . The choices of the v_{2j} are otherwise arbitrary, except for that $\mathcal{P}(i, K) := \{P(x, y) : \{x, y\} \in E(i, K)\}$ has to consist of pairwise disjoint paths.

We can indeed choose the preimages $P(x, y)$ to be pairwise disjoint and fulfill (III), by the choice of the capacity constraints in $V(\tilde{K})$: every $x \in V(\tilde{K})$ is crossed by at most as many paths of $\tilde{\mathcal{P}}$, as the number of edges of M_{i-1} with a yellow (respectively: blue) endpoint in $\tilde{\beta}^{-1}(x)$, and so there is a choice when each of these edges is present in at most one of the augmenting paths. ■

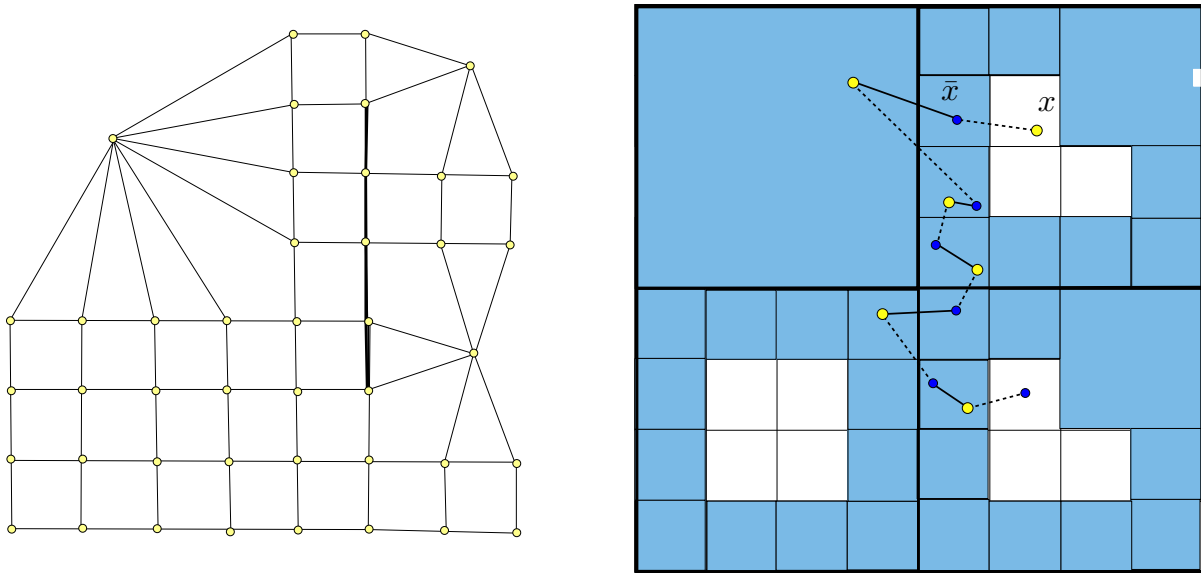


Figure 4. A path of $\tilde{\mathcal{P}}(i, K)$ in \tilde{K} (left), and the corresponding augmenting path of $\mathcal{P}(i, K)$ in K (from Figure 3). The shaded region is $\tilde{\beta}^{-1}\partial\beta\mathcal{P}_{b(i)}$.

We shall consider the path $P(x, y)$ as an *augmenting path*, and replace the edges of $P(x, y)$ in \mathcal{M}_{i-1} by those that are not in there. Doing this over all $K \in \mathcal{P}_{a(i)}$ and $P(x, y)$, we get \mathcal{M}_i . For a particular $K \in \mathcal{P}_{a(i)}$, after doing the “flip” for each augmenting path in \mathcal{P} , all of K but $K \cap U_i$ is matched. Moreover, the densities of the different edge-lengths still decay exponentially, see Lemma 3.13.

Let us summarize our conclusion about augmenting paths, also adding a claim that the limiting perfect matching exists (which is yet to be shown):

PROPOSITION 3.12. *If the conditions (and hence the conclusions) of Proposition 3.11 hold, then **Step 2** is successful, and by the end of stage i , all vertices in $K \setminus U_i$ are matched. As $i \rightarrow \infty$, we get a perfect matching in the limit.*

The following lemma tells that the matchings \mathcal{M}_j stabilize (hence proving the last assertion of Proposition 3.12), and gives the tail probabilities for the limiting matching. For simplicity, in this statement and its proof, by an edge we mean a pair in one of the \mathcal{M}_i 's. For a vertex x , $\mathcal{M}_i(x)$ will denote the pair of x by \mathcal{M}_i , or the emptyset if x is not matched. By $\|e\|$ we denote the distance between the two endpoints of e .

LEMMA 3.13. *Suppose the conditions in Proposition 3.11 are satisfied. Then for every vertex $v \in \mathbb{Z}^d$ there is a number $j(v)$ such that v is contained in an edge e with the property that $e \in \mathcal{M}_j$, for every $j \geq j(v)$. Moreover, we have*

$$\mathbf{P}[\|e\| > r] \leq c \exp(-c' r^{d-2-\epsilon}).$$

Proof. The edges stabilize for the following reason. In order for a vertex x to be in different edges or no edge in $(\mathcal{M}_j)_j$ infinitely many times, it is necessary that either x or $\mathcal{M}_j(x)$ is contained in $\cup_{K \in \mathcal{P}_{a(j)}} \mathcal{P}(j, K)$, and hence in $\tilde{\beta}^{-1} \partial \tilde{\beta} \tilde{\mathcal{P}}_{b(j)} \cup U_j$, for infinitely many j 's. The probability of this is 0, since $\sum \mathbf{P}[x \in \tilde{\beta}^{-1} \partial \tilde{\beta} \tilde{\mathcal{P}}_{b(j)} \cup U_j] + \mathbf{P}[\mathcal{M}_j(x) \in \tilde{\beta}^{-1} \partial \tilde{\beta} \tilde{\mathcal{P}}_{b(j)} \cup U_j] \leq 2 \sum \mathbf{P}[x \in \tilde{\beta}^{-1} \partial \tilde{\beta} \tilde{\mathcal{P}}_{b(j)} \cup U_j]$ by Lemma 1.9, and this sum is finite.

Now, let e be as in the claim, and $j(v)$ be the smallest nonnegative integer such that $e \in \mathcal{M}_j$ whenever $j \geq j(v)$. If $j(v) = 0$, then by definition of \mathcal{M}_0 , the endpoints of e are in the same \mathcal{P} -cell, and the claim follows by Lemma 3.10.

Otherwise there is some sequence of edges that v is contained in, until it stabilizes from $\mathcal{M}_{j(v)}$ on. Let h be the greatest number such that $v \in U_h$ if such a U_h exists, otherwise $h := 0$. There is an m , and a sequence $i_1, \dots, i_m, i_1 := h+1$, such that $\mathcal{M}_{i_j}(v) \neq \mathcal{M}_{i_{j-1}}(v)$. We may assume that m is maximal such. Let the edge containing v in \mathcal{M}_{i_j} be called e_j . By maximality, $e_m = e$. Since $v \in U_{i_1-1} = U_h$, by (II) and (III) we have that $\mathcal{M}_{i_1}(v)$ is in the $\tilde{\beta}^{-1} \mathcal{P}_{b(i_1)}$ -cell of v . Thus $\|e_1\| \leq \sqrt{d} 2^{b(i_1)+1}$, unless v is in a \mathcal{P} -cell of size $> 2^{b(i_1)}$. This shows the following tail for $\|e_1\|$:

$$\begin{aligned} \mathbf{P}[\|e_1\| > r] &\leq 2\mathbf{P}[v \in U_j, j \geq \log \frac{r}{2\sqrt{d}}] + \mathbf{P}[\mathcal{P}(v) \text{ has radius} \geq r/2d^{1/2}] \leq \\ &\leq c2^{-a([\log r/2\sqrt{d}])^{1+\epsilon/2}} + c_0 \exp(-c'_0 r^{d-2-\epsilon}) \leq c \exp(-c' r^{d-2-\epsilon}). \end{aligned} \quad (3.14)$$

We got the second term on the left of (3.14) by Lemma 3.6, and the first term from

$$\begin{aligned} \mathbf{P}[v \in U_i, j \geq \log r/2\sqrt{d}] &\leq c \sum_{j=[\log r/\sqrt{d}]}^{\infty} \min_{x,y \in U_i} \text{dist}(x,y)^{-d} \leq \\ &\leq \sum_{j=[\log r/\sqrt{d}]}^{\infty} 2^{-a(j)^{1+\epsilon/2}} \leq c2^{-a([\log r])^{1+\epsilon/2}}, \end{aligned}$$

using (3.8). For the last inequality of (3.14), we used the choice of $a(i)$.

We get each e_i from e_{i-1} by the “switch” along some augmenting path $P(x, y)$, and (by the choice of h) we also assumed that they are inner edges of this path, whenever $i - 1 > 1$. Thus, e_i and e_{i-1} being consecutive edges in $P(x, y)$, they share one endpoint, while their other endpoints ($x(e_i)$ and $x(e_{i-1})$) are in adjacent \mathcal{P} -cells). Hence $\mathbf{P}[||e_i|| - ||e_{i-1}|| \geq r] \leq \mathbf{P}[|x(e_j) - x(e_{j-1})| \geq r] \leq 4\mathbf{P}[\mathcal{P}(o)$ has diameter $\geq r/4] \leq c \exp(-c'(r)^{d-2-\epsilon})$, by Lemma 1.9 and Lemma 3.6. That is,

$$\mathbf{P}[||e_i|| - ||e_{i-1}|| \geq r] \leq c \exp(-c'r^{d-2-\epsilon}). \quad (3.15)$$

Also, for each edge e_i , if v switched from edge e_{i-1} to edge e_i in some stage j , then in particular, one of the endpoints of e_i has to be in $\tilde{\beta}^{-1}\partial\tilde{\beta}\mathcal{P}_{b(j)}$. We can give the following rough bound on this probability:

$$\begin{aligned} 2\mathbf{P}[o \in \tilde{\beta}^{-1}(\partial\tilde{\beta}\mathcal{P}_{b(j)})] &= 2\mathbf{P}[\tilde{\beta}\mathcal{P}_{b(j)}(o) \in \partial\tilde{\beta}(\mathcal{P}_{b(j)})] \leq \\ &\leq \mathbf{P}[\text{diam}(\mathcal{P}(o)) \leq b(j), \text{dist}(o, \partial(\mathcal{P}_{b(j)}(o))) \leq b(j)] + \mathbf{P}[\text{diam}(\mathcal{P}(o)) \geq b(j)] \leq \\ &\leq \mathbf{P}[\text{dist}(o, \partial(\mathcal{P}_{b(j)}(o))) \leq b(j)] + \mathbf{P}[\text{diam}(\mathcal{P}(o)) \geq b(j)] \leq \\ &\leq c2^{-b(j)}b(j) + c \exp(-c'b(j)^{d-2-\epsilon}) \leq c \exp(-c'b(j)^{d-2-\epsilon}). \end{aligned} \quad (3.16)$$

Here the bound on the first probability is a consequence of Lemma 1.8, and the bound on the second probability is by Lemma 3.6.

Let A be the event that $m > \log R$ and B be the event that $\{ \{ ||e_0|| > R/\log R \} \text{ or } ||e_i|| - ||e_{i-1}|| > R/\log R \text{ for some } i \in \{2, \dots, m\} \}$. We finish the proof by noting that for $|e| > R$ to happen, one has to have at least one of A and B hold. Using (3.16) to bound the probability of A (which is $\leq 2\mathbf{P}[o \in \cup_{j=\lceil \log r \rceil}^{\infty} \tilde{\beta}^{-1}\partial\tilde{\beta}(\mathcal{P}_{b(j)})]$), and (3.14) and (3.15) to show $\mathbf{P}[B] \leq c \log R \exp(-c'(R/\log R)^{d-2-\epsilon})$, this gives

$$\mathbf{P}[e > R] \leq c \exp(-c'b(\lceil \log R \rceil)) + c \exp(-c'(R/\log R)^{d-2-\epsilon}) \log R.$$

We conclude that

$$\mathbf{P}[e > R] \leq c \exp(-c'R^{d-2-2\epsilon}),$$

since $b(\lceil \log R \rceil) \geq R^d$. ■

So, all what is left is to show the existence of an admissible flow in $N(\tilde{K})$ of strength $\sigma = \frac{|U_{i-1}| + |U_i|}{2}$ from B to Y , since then the conditions in Proposition 3.12 and Lemma 3.13 follow.

By the maxflow-mincut theorem, the existence of such a flow follows if we show that every mincut has capacity $\geq \sigma$.

Before proving this, let us state a rough estimate relating the size of the boundary of a subgraph of \tilde{K} (or K) to the number of dyadic pseudocubes of a certain type in it. Given $A \subset K$, denote by ∂A as before, the set of vertices that have degree $< 2d$ in A , and denote by $\partial_j A$ the j -neighborhood of ∂A .

LEMMA 3.17. *Let \tilde{A} be an induced subgraph of \tilde{K} , and $A := \tilde{\beta}^{-1}(\tilde{A}) \subset K$. Then, for the set $D_\ell(A)$ of dyadic $k2^\ell$ -pseudocubes in A that are not contained in any larger dyadic pseudocube in A , we have*

$$|D_\ell(A)| \leq 2|\partial A|d^{1/2}(k2^\ell)^{1-d}.$$

Proof. Every element of $D_\ell(A)$ is contained in $A \cap \partial_{d^{1/2}k2^{\ell+1}} A$, as can be seen by induction on i . The elements of $D_\ell(A)$ are disjoint. Thus dividing $|A \cap \partial_{d^{1/2}k2^{\ell+1}} A|$ by the lower bound $(k2^{\ell-1})^d$ on the volume of an element in D_ℓ , we get an upper bound $2d^{1/2}|\partial A|(k2^\ell)^{1-d}$ for $D_\ell(A)$. Here we were also using $|A \cap \partial_{d^{1/2}k2^{\ell+1}} A| \leq |\partial A|d^{1/2}k2^{\ell+1}$. \blacksquare

PROPOSITION 3.18. *There is an admissible flow of strength σ from B to Y in $N(\tilde{K})$.*

Proof. We will prove by contradiction. So, suppose that there is some π minimal edge-cutset between B and Y , and that $\text{cap}(\pi) < \sigma = \sum_x \text{cap}(\{B, x\}) = \sum_x \text{cap}(\{Y, x\})$. Hence π contains some edge of \tilde{K} . Let $\gamma = \pi \cap \tilde{K}$, and C_1, C_2, \dots, C_m be the connected components of $(\tilde{K} \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}) \setminus \gamma$. One can find a set γ' of edges of 0 capacity, such that $\tilde{K} \setminus (\gamma \cup \gamma')$ has components C'_1, \dots, C'_m , such that that $C_i \subset C'_i$, and γ' minimal with this property (i.e., for any $e \in \gamma \cup \gamma'$, $\tilde{K} \setminus (\gamma \cup \gamma') \cup \{e\}$ has less than m components). Furthermore, one has:

$$|\gamma'| \leq 2^{d(b(i)+1)+1} |\gamma \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}|. \quad (3.19)$$

This is true because by definition, every cell C of $\mathcal{P}_{b(i)}$ is either a pseudocube of size $2^{b(i)}$ in $\mathcal{Q}_{b(i)}$, or an elementary cell from \mathcal{P} (which is thus contracted by $\tilde{\beta}$), hence every class of $\tilde{\beta}\mathcal{P}_{b(i)}$ has cardinality $\leq 2^{db(i)+1}$. We can assign to each element of $\gamma \cap C$, $C \in \tilde{\beta}\mathcal{P}_{b(i)}$, some $e \in \partial\tilde{\beta}\mathcal{P}_{b(i)}$ in the $\tilde{\beta}\mathcal{P}_{b(i)}$ -cell of C , to get (3.19).

Note that the edges in γ' have costs 0 by the definition of $N(\tilde{K})$. Hence we may replace π by $\pi \cup \gamma'$ and γ by γ' , to have (by (3.19)):

LEMMA 3.20. *There exists a minimal cutset π between B and Y in $N(\tilde{K})$, $\gamma = \tilde{K} \cap \pi$, such that the number of components in $\tilde{K} \setminus \gamma$ is the same as the number of components in $(\tilde{K} \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}) \setminus \gamma$, and further, $|\gamma| \leq (2^{d(b(i)+1)+1} + 1)|\gamma \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}|$.*

Now, for each C_j , one of N_{Y, C_j} and N_{B, C_j} (call it N_j) has to belong to π , otherwise there is a path from B to Y through C_j that avoids π .

LEMMA 3.21. *There is a $j' \in \mathbb{Z}$ such that*

$$\sum_{j \neq j'} |\partial(\tilde{\beta}^{-1}C_j)| \leq c|\tilde{\beta}^{-1}\gamma| \leq c|\gamma| \leq c'2^{b(i)d}|\gamma \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}|.$$

Proof. The first inequality is by Lemma 3.3. The second is simply because after contracting by $\tilde{\beta}$ we kept multiple edges. The last, rough inequality is true because of Lemma 3.20. ■

Apply Lemma 3.3 to $\Gamma := \tilde{\beta}^{-1}(\gamma)$ and $\Delta_j := \tilde{\beta}^{-1}(C_j)$. We may assume that $i = 1$ is the (possible) exception in Lemma 3.3 (and equivalently that C_1 is the exception in Lemma 3.21), and assume by symmetry that $N_1 = N_{B,C_1}$. Furthermore, we may assume that every Δ_j is such that

$$|\Delta_j \cap U_i| > 1, \quad (3.22)$$

otherwise if $\Delta_j \cap U_i = \{x\}$, it is easy to see that we could remove part of ∂C_j (edges of capacity $k^{\frac{d}{d-1}}/12d' > 1$) from π , and add the edge between \bar{x} and B or Y , to get a cutset of smaller cost than π . Similar argument works if $\Delta_j \cap U_i = \emptyset$.

As mentioned, if π is a minimal cutset, then

$$\text{cap}(\pi) \leq \text{cap}(N_{B,\tilde{K}}), \quad (3.23)$$

since $N_{B,\tilde{K}}$ is a cutset itself.

Use notation $C(b(i)) := (c2^{b(i)d})^{-1}k^{d/(d-1)}/(12d')$, where the first factor and the constant c there is coming from Lemma 3.21, while the second factor is the capacity of edges in $\gamma \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}$. By Lemma 3.21, using that every edge of $\gamma \cap \partial\tilde{\beta}\mathcal{P}_{b(i)}$ has capacity $k^{d/(d-1)}/(12d')$, we can bound the left hand side of (3.23) as

$$\begin{aligned} \text{cap}(\pi) &= \text{cap}(\gamma) + \sum_{j=1}^m \text{cap}(N_j) \geq C(b(i))|\gamma| + \sum_{j=1}^m \text{cap}(N_j) = \\ &= C(b(i))|\gamma| + \text{cap}(N_1) + \sum_{j=2}^m \text{cap}(N_{B,C_j}) - \mathbf{1}_{N_j=N_{Y,C_j}} (\text{cap}(N_{B,C_j}) - \text{cap}(N_{Y,C_j})) \geq \\ &\geq C(b(i))|\gamma| + \text{cap}(N_{B,C_1}) + \sum_{j=2}^m \text{cap}(N_{B,C_j}) - \mathbf{1}_{N_j=N_{Y,C_j}} \text{sur}(\Delta_j \cap U_{i-1}) - |\Delta_j \cap U_i|. \end{aligned}$$

Note that we were using our assumption on $C_1 = \tilde{\beta}(\Delta_1)$ for the last inequality. Putting this fact together with the last inequality and (3.23), we obtain

$$\text{cap}(N_{B,\tilde{K}}) \geq C(b(i))|\gamma| + \sum_{j=1}^m \text{cap}(N_{B,C_j}) - \sum_{j=2}^m (\mathbf{1}_{N_j=N_{Y,C_j}} \text{sur}(\Delta_j \cap U_{i-1}) + |\Delta_j \cap U_i|).$$

(We define the empty sum as 0. That corresponds to the case when γ is not a cutset for \tilde{K} .) This implies

$$\sum_{j=2}^m |\Delta_j \cap U_i| + \sum_{j=2}^m \text{sur}(\Delta_j \cap U_{i-1}) \geq C(b(i))|\gamma|. \quad (3.24)$$

On the other hand, the minimal distance between elements of U_{i-1} is $\geq 2^{\frac{a(i-1)(1+\epsilon/2)}{d}}$ by the choice of U_{i-1} (see (3.8)). Let $A(i) := 2^{a(i)(1+\epsilon/2)/d}$. We have that

$$\begin{aligned} \text{sur}(\partial_{A(i-1)}\Delta_j \cap U_{i-1}) &\leq |\partial_{A(i-1)}\Delta_j \cap U_{i-1}| \leq |\{x : B_{A(i-i)}(x) \subset \partial_{2A(i-1)}\Delta_j\} \cap U_{i-1}| \leq \\ &\leq |\partial_{2A(i-1)}\Delta_j|/A(i-1)^d \leq c|\partial\Delta_j|/A(i-1)^{d-1}. \end{aligned} \quad (3.25)$$

Using notation D_ℓ coming from Lemma 3.17, observe that

$$\Delta_j \subset \partial_{A(i-1)}\Delta_j \cup \left(\bigcup_{\ell=\log(A(i-1))}^{a(i)} \bigcup_{X \in D_\ell(\Delta_j)} X\right).$$

Subadditivity $\text{sur}(H \cup H') \leq \text{sur}(H) + \text{sur}(H')$ yields

$$\begin{aligned} \text{sur}(\Delta_j \cap U_{i-1}) &\leq \text{sur}(\partial_{A(i-1)}\Delta_j \cap U_{i-1}) + \sum_{\ell=\log(A(i-1))}^{a(i)} \sum_{X \in D_\ell(\Delta_j)} \text{sur}(X \cap U_{i-1}) \leq \quad (3.26) \\ &\leq c|\partial\Delta_j|A(i-1)^{1-d} + \sum_{\ell=\log(A(i-1))}^{a(i)} |D_\ell(\Delta_j)|f((k2^\ell)^d). \end{aligned}$$

Here we used (3.25) for the first term, and for the second term we used the fact that dyadic pseudocubes are not bad by definition.

For the first term in (3.24), we get the following bound. In the first inequality we use Lemma 3.9 and (3.22), and in the second one we use Lemma 3.3 and the choice of C_1 :

$$\sum_{j=2}^m |\Delta_j \cap U_i| \leq c2^{-a(i)\epsilon/2} \sum_{j=2}^m |\partial\Delta_j| \leq c2^{-a(i)\epsilon/2}|\gamma|.$$

Plugging this and (3.26) into (3.24), we obtain

$$\begin{aligned} C(b(i))|\gamma| &\leq c2^{-a(i)\epsilon/2}|\gamma| + \sum_{j=2}^m \left(c|\partial\Delta_j|A(i-1)^{1-d} + \sum_{\ell=\log(A(i-1))}^{a(i)} |D_\ell(\Delta_j)|f((k2^\ell)^d) \right) \leq \\ &\leq c2^{-a(i)\epsilon/2}|\gamma| + \sum_{j=2}^m |\partial\Delta_j| \left(cA(i-1)^{1-d} + \sum_{\ell=\log(A(i-1))}^{a(i)} (k2^\ell)^{1-d} f((k2^\ell)^d) \right) \leq \\ &\leq |\gamma| \left(c + cA(i-1)^{1-d} + \sum_{\ell=\log(A(i-1))}^{a(i)} (k2^\ell)^{1-d} (k2^\ell)^{d-1-\epsilon/2} \right). \end{aligned} \quad (3.27)$$

The penultimate inequality was a consequence of Lemma 3.17 applied to each Δ_j , while the last line follows from the first two inequalities of Lemma 3.21.

We conclude, by the definition of f and $C(b(i))$, and using $\log(A(i-1)) \geq a(i-1)/d$:

PROPOSITION 3.28. *If π is some cutset between B and Y , $\gamma = \tilde{K} \cap \pi$, and $\text{cap}(\pi) < \min\{\text{cap}(N_{B, \tilde{K}}), \text{cap}(N_{Y, \tilde{K}})\}$, then*

$$k^{\frac{d}{d-1}} b(i)^{-d} \leq c + C 2^{(-1+1/d)a(i-1)(1+\epsilon/2)} + \sum_{\ell=a(i-1)/d}^{a(i)} (k 2^\ell)^{-\epsilon/2}, \quad (3.29)$$

where c and C are constants that depend only on d .

The second term on the right is bounded by a constant (independently of k). By the definitions of $a(i)$ and $b(i)$, for $i \geq 2$, the inequality (3.29) fails, so there is no minimal cutset between B and Y different from $N_{B, \tilde{K}}$ or $N_{Y, \tilde{K}}$, showing that the desired flow on $N(\omega)$ exists. If $i = 1$, if k was chosen large enough, then (3.29) fails. This finishes the proof of Proposition 3.18. \blacksquare

From Proposition 3.18 and Proposition 3.11 we have the existence of $\mathcal{P}(i, K)$, and conclude by Proposition 3.12 that the rematching procedure succeeds. Combined with Lemma 3.13, this proves Theorem 1.2.

We mention that once the $a(i)$ and $b(i)$ can be chosen to grow with a suitable speed, the crucial inequalities above can be made to be true *for any* $i \geq 1$. The major difficulty is the start, since $a(0)$ and $b(0)$ has to be set 1. This is responsible for the isoperimetric considerations and “tightness” in the above computations.

REMARK 3.30. Note that $f(x)$ could be chosen any other way so that (3.29) fails, and the tail for the matching is given by Lemma 3.10. All one has to ensure is that the sum on the right of (3.29) resulting with this new f is finite when summed up to infinity. E.g. $f(x) := x^{\frac{d-1}{d}} (\log x \log \log x)^{-1}$ gives the bound $\mathbf{P}[o \text{ is matched to distance } \geq r] \leq \exp(\frac{-cr^{d-2}}{\log r \log \log r})$. We have chosen the slightly weaker bound to make the formulas easier to follow.

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